THE PROPAGATION OF AN ELASTIC-PLASTIC WAVE IN A MATERIAL WITH DISLOCATION KINETICS OF PLASTIC DEFORMATION

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The propagation of elastic-plastic waves in a solid, initiated under an impact load, is dependent on the type of material, i.e., its rheological, relaxational, dissipative, and dispersion properties, which may depend nonlinearly on the magnitude of the applied impact stress. In one of the first analytical studies of the processes involved in the generation of waves [1] under conditions of one-dimensional deformation the authors resorted to the comparatively simple determining equation for a viscoelastic body with quadratic nonlinearity. An interesting result was presented in [1], which is, as yet, not completely clear from the physical standpoint, and according to which the time required for the wave to enter a steady-state propagation regime is approximately greater by a factor of five than the duration of the steady wave front. However, this result is not in agreement with experimental data obtained for elastic-viscoplastic materials. In this connection, it is extremely important that an analogous analysis be undertaken, while maintaining the one-dimensionality of the problem, for the more complex form of the determining equation, and in addition the derived results must be compared against experimental data. One of these realistic determining equations, widely discussed in the literature, is the Sokolovskii–Malverne equation and its dislocation variant [2, 3].

In the present study we undertake an analytical investigation into the process of the propagation of a nonsteady elastic-plastic wave in a material with dislocation kinetics of plastic deformation. We have set ourselves the goal of deriving an analytical relationship for the so-called Bland number [1], i.e., the ratio of the time for the entry of the wave onto the steady-state propagation segment relative to the duration of its steady front. Knowledge of this characteristic makes it possible to estimate the time required to generate waves in various materials, without conducting any complex and expensive experiment.

1. Formulation of the Problem. We solve a system of equations for a continuous medium, this system closed by an equation in dislocation form:

$$\rho \partial^2 u_i/\partial t^2 + \sigma_{ij,j} = 0; \tag{1.1a}$$

$$\partial \varepsilon_{ij}/\partial t + \partial^2 u_i/\partial t\, \partial x_j = 0; \tag{1.1b}$$

$$\sigma_{ij} = F[\varepsilon_{ij}, \dot{\varepsilon}_{ij}, \ldots]. \tag{1.1c}$$

To obtain the uniaxial load with the aid of transition to shear strain γ (see [2]) we will rewrite system (1.1a)-(1.1c) in one-dimensional form. The determining equation in dislocation form is found in [2]. In conclusion

$$\rho u_{tt} + \sigma_x = 0; \tag{1.2a}$$

$$\varepsilon_t + u_{tx} = 0; \tag{1.2b}$$

$$\sigma_t - \rho c^2 \varepsilon_t = -\frac{8}{3}\mu \frac{b^2 N_0}{B}\left(1 + \frac{\alpha}{N_0}\gamma\right)(\tau - \tau_i), \tag{1.2c}$$

where ρ is the density of the material; u, the displacement of the particles in the medium; σ, the normal stress; ε, the total strain (the elastic strain in combination with the plastic strain) in the direction of wave propagation; c, the longitudinal speed of sound; b, the value of the Burgers vector; N_0, the initial dislocation density; τ, the maximum resolving stress; τ_i, the inverse stress [2]; α, the dislocation multiplication factor. The subscripts t and x indicate differentiation with respect to time and coordinate, while the symbol beyond the comma at the tensor reflects differentiation with respect to the corresponding coordinate.

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With uniaxial deformation of isotropic materials τ and γ can be expressed in terms of the normal components of total strain and stress:

$$\tau = (3/4)[\sigma - (\lambda + 2\mu/3)\varepsilon]; \tag{1.3}$$

$$\gamma = (3\mu/8)[\sigma - (\lambda + 2\mu)\varepsilon]. \tag{1.4}$$

Having eliminated the normal stress from (1.3), we have $\tau = \mu(\varepsilon - 2\gamma)$. Then the determining equation (1.2c) assumes the form

$$\gamma_t = \gamma_*(1 + M\gamma)(\varepsilon - 2\gamma), \tag{1.5}$$

where $\gamma_* = b^2N_0\mu/B$, $M = \alpha/N_0$. System of equations (1.2a), (1.2b), (1.5) is reduced to a single nonlinear equation for the shear strain γ:

$$\Box_c\left(\frac{\gamma_t}{\gamma_*(1+M\gamma)}\right) + \frac{2c_h^2}{c^2}\Box_{c_h}\gamma = 0 \tag{1.6}$$

($\Box_a \equiv \partial^2/\partial x^2 - (1/a^2)\partial^2/\partial t^2$ is the standard notation for the wave operator, $c_h^2 = c^2\left(1 - \frac{4}{3}\frac{\mu}{\rho c^2}\right)$ is the hydrostatic velocity). Equation (1.6) is enhanced with the initial and boundary conditions. For the case of uniaxial stepwise loading to displace the particles of the medium, we will write these initial and boundary conditions in the following form:

$$u(x,\ 0) = u_t(x,\ 0) = 0; \tag{1.7a}$$

$$u_t(0,\ t) = v_0\theta(t) \tag{1.7b}$$

[$\theta(t)$ is the Heaviside function]. We will rewrite these conditions to the condition at γ, using the Seitz–Reed relationship (assuming that it is valid at the boundary of the target):

$$\gamma_t^{(0)} = bN_{m0}v_d^{(0)} \tag{1.8}$$

($\gamma_t^{(0)}$, N_{m0}, $v_d^{(0)}$ respectively denote the shear strain, the density, and the velocity of the dislocations at the boundary of the target). The dislocation velocity is expressed in terms of the stress

$$v_d = (\tau - \tau_i)b/B, \tag{1.9}$$

the maximum stress is replaced by the normal stress: $\tau = \sigma g$, where $g = (1 - 2\nu)/[2(1 - \nu)]$, ν is the Poisson coefficient. In the subsequent derivation it is essential that we take into consideration the Hugoniot–Rankine relationship at the boundary of the material:

$$\sigma = \rho c u_t^{(0)}. \tag{1.10}$$

Using (1.7a)-(1.10), and also in the assumption of a purely elastic response from the material at the initial instant of time, we will express the initial and boundary conditions at γ as follows:

$$\gamma_x(0,\ t) = -(1/c)\gamma_*\left[\rho c u_t^{(0)}(t)g/\mu - \tau_*\right];$$
$$\gamma_t(0,\ t) = \gamma_*\left[g\rho c u_t^{(0)}(t)/\mu - \tau_*\right];$$
$$\gamma_t(x,\ 0) = \gamma_*\left[g\rho c u_t^{(0)}(0)/\mu - \tau_*\right];$$
$$\gamma(x,\ 0) = 0;\quad \tau_* = \tau_0/\mu.$$

2. Method of Solution. Let us note that as $M \to \infty$ in Eq. (1.6) it is possible to drop the first term, and in conclusion we are left with an ordinary wave equation for γ, which in the form of a solution provides a simple wave moving at hydrostatic velocity. This result is analogous to one of the conclusions in [4].

In Eq. (1.6), in order to find the solution, the small parameter κ is isolated, and for this reason it is rewritten in the convenient form

$$\Box_{c_h}\left(\gamma + \beta\frac{\gamma_t}{1/M + \gamma}\right) + \varkappa\frac{\beta}{c^2}\left(\frac{\gamma_t}{1/M + \gamma}\right)_{tt} = 0 \tag{2.1}$$

$$\left(\beta \equiv \frac{c^2}{2c_h^2 M\gamma_*}, \quad \varkappa \equiv 1 - c_h^2/c^2\right).$$

Since $c_h \sim c$, κ is obviously dimensionless and consequently, in zeroth approximation, the term containing κ can be dropped. In this case Eq. (2.1) assumes the form

$$\Box_{c_h} \Xi = 0, \tag{2.2}$$

where

$$\Xi(x, t) = \gamma(x, t) + \beta \frac{\gamma_t(x, t)}{1/M + \gamma(x, t)}. \tag{2.3}$$

The boundary and initial conditions at γ are easily expressed in terms of the function $\Xi(x, t)$:

$$\Xi(0, t) = (A - B)t + \beta \frac{A\theta(t) - B}{1/M + (A - B)t}; \tag{2.4a}$$

$$\Xi_x(0, t) = -\frac{1}{c}(A\theta(t) - B) + \beta \frac{(A\theta(t) - B)^2}{c[1/M + (A - B)t]^2} - \beta \frac{A\delta(t)}{1/M + (A - B)t}; \tag{2.4b}$$

$$\Xi(x, 0) = \beta M(A - B)\theta(-x^2) \tag{2.4c}$$

$$(A = \gamma_* g\rho c v_0/\mu, \; B = \gamma_* \tau_*).$$

Solution of problem (2.2), (2.4a)-(2.4c) is sought in the form

$$\Xi(x, t) = V(x, t) + W(x, t),$$

where under the condition $W(x, 0) = \Xi(x, 0)$

$$W(x, t) = \beta M(A - B)\theta(c_h^2 t^2 - x^2).$$

For the function $V(x, t)$ we obtain the system

$$\begin{aligned} &\Box_{c_h} V = 0, \quad V(x, 0) = 0, \\ &V(0, t) = \Xi(0, t) - \beta M(A - B) \equiv u_0(t), \\ &V_x(0, t) = \Xi_x(0, t) \equiv u_1(t). \end{aligned} \tag{2.5}$$

The solution for this problem will be

$$V(x, t) = g_x(x, \cdot) * u_0(\cdot) + g(x, \cdot) * u_1(\cdot). \tag{2.6}$$

Convolution in formula (2.6) is accomplished by means of the Green's function:

$$g(x, t) = \frac{c_h}{2}\theta\left(\frac{x^2}{c_h^2} - t^2\right).$$

To obtain the analytical form of the solution for problem (2.5), we choose a region in which the velocity of motion for the plastic wave changes. It is assumed that its velocity cannot be smaller than the hydrostatic velocity c_h and larger than the elastic velocity c ($t < x/c_h$). Using this condition, we can rewrite (2.5) to the form

$$V(x, t) = \frac{1}{2}\left[u_0\left(\frac{x}{c_h} + t\right) - u_0\left(\frac{x}{c_h} - t\right)\right] + \frac{c_h}{2}\int_{x/c_h - t}^{x/c_h + t} u_1(\tau)\, d\tau.$$

Then, for the function $\Xi(x, t)$, with consideration of the specific form of the boundary and initial conditions, we finally obtain

$$\Xi(x, t) = \beta M(A - B)\theta(c_h^2 t^2 - x^2) + (A - B)\left(1 - \frac{c_h}{c}\right)t +$$

$$+\frac{\beta}{2}\left[\frac{A\theta\left(\frac{x}{c_h}+t\right)-B}{1/M+(A-B)\left(\frac{x}{c_h}+t\right)}-\frac{A\theta\left(\frac{x}{c_h}-t\right)-B}{1/M+(A-B)\left(\frac{x}{c_h}-t\right)}\right]+$$
$$+\beta(c_h/c)\,t/(\{1/[M(A-B)]+x/c_h\}^2-t^2).$$

We will now solve the ordinary differential equation (2.3), substituting $y(x, t) = 1/(1/M + \gamma(x, t))$. As a result,

$$\gamma(x,t)=\frac{\exp\left[\frac{1}{\beta}\int\limits_{x/c_p}^{t}\left(\Xi(x,\tau)+\frac{1}{M}\right)d\tau\right]}{\eta+\frac{1}{\beta}\int\limits_{x/c_p}^{t}\exp\left[\frac{1}{\beta}\int\limits_{x/c_p}^{\tau}\left(\Xi(x,z)+\frac{1}{M}\right)dz\right]d\tau}-\frac{1}{M}. \tag{2.7}$$

The integration constant η is found from the condition of equality to zero for the shear strain at the initial instant of time:

$$\eta=M\exp\left[\frac{1}{\beta}\int\limits_{x/c_p}^{0}\left(\Xi(x,\tau)+\frac{1}{M}\right)d\tau\right]-\frac{1}{\beta}\int\limits_{x/c_p}^{0}\exp\left[\frac{1}{\beta}\int\limits_{x/c_p}^{\tau}\left(\Xi(x,z)+\frac{1}{M}\right)dz\right]d\tau.$$

Integrating over the exponential, for

$$F(x,t)=\frac{1}{\beta}\int\limits_{x/c_p}^{t}\left(\Xi(x,\tau)+\frac{1}{M}\right)d\tau$$

we obtain

$$F(x,t)=\frac{1}{\beta}\left[\frac{1}{M}\left(t-\frac{x}{c_p}\right)+\frac{A-B}{2}\left(1-\frac{c_h}{c}\right)\left(t^2-x^2/c_p^2\right)\right]+$$
$$+\left(1-\frac{c_h}{c}\right)\ln\left(1-\frac{t^2-x^2/c_p^2}{\left[\frac{1}{M(A-B)}+\frac{x}{c_h}\right]^2-x^2/c_p^2}\right). \tag{2.8}$$

<u>3. Entry of the Wave into the Steady State.</u> As the wave velocity approaches the steady state, the logarithm in expression (2.8) can be expanded, bearing in mind the smallness of the term removed from under the logarithm sign: $F(x, t) = \omega(x, t)(t - x/c_p)$, where

$$\omega(x,t)=1/M\beta+a(K-\xi(x))(t+x/c_p). \tag{3.1}$$

Here $a = 1 - c_h/c$; $K = (A - B)/(2\beta)$; $\xi(x)=\left[\left(\frac{1}{M(A-B)}+\frac{x}{c_h}\right)^2-x^2/c_p^2\right]^{-1}$. Subsequently, the function $\omega(x, t)$ is studied to the limit, and the physical critical point is chosen, for which

$$x_0=\frac{(M(A-B))^{-1}}{c_h\left(1-c_h^2/c_p^2\right)}\left(\sqrt{1+\left(1-c_h^2/c_p^2\right)c_h^2\left(\frac{M^2(A-B)^2}{K}-1\right)}+1\right); \tag{3.2a}$$
$$t_0=\frac{(M(A-B))^{-1}}{c_hc_p\left(1-c_h^2/c_p^2\right)}\left(\sqrt{1+\left(1-c_h^2/c_p^2\right)c_h^2\left(\frac{M^2(A-B)^2}{K}-1\right)}+1\right). \tag{3.2b}$$

In these calculations we take into consideration the form of the relationship between the ordinary wave and the coordinates and time.

The characteristic equation of quadratic form for the second derivatives of the function $\omega(x, t)$ has the form

$$\lambda(\lambda + (2a/c_p)\xi_x(x_0)) + a\xi_x(x_0) = 0,\ \xi_x(x_0) < 0,$$

whose roots are as follows:

$$\lambda_{1,2} = \frac{a}{c_p} |\xi_x(x_0)| \left(1 \pm \sqrt{1 + \frac{c_p^2}{a|\xi_x(x_0)|}}\right).$$

It is obvious that the found critical point is the point of inflection. By means of formula (3.1) it is not difficult to prove that ω(x, t) is a diminishing function. In the following it is assumed that the bending segment in which the frequency changes by no more than 10% in either direction is that section in which the propagation of the wave is steady, so that t_0 represents the time required for the wave to reach the steady state. As $c_p \to c_h$ the root in expression (3.2b) can be expanded in series, leading to

$$t_0 = \frac{2\,(c_h/cp)}{M\gamma_*(g\rho c v_0/\mu - \tau_*)\left(1 - c_h^2/c_p^2\right)}.$$

This yields an interesting result, according to which the time required for the wave to reach the steady-state segment of propagation is reduced as the loading velocity increases in conjunction with the density of the loaded material, and it increases as the velocity of plastic-wave propagation approaches the hydrostatic velocity.

In (2.7) the integral in the denominator is evaluated as follows:

$$\int_0^t \exp[F(x, \tau)]\, d\tau \approx t \exp[F(x, t)],$$

and the error of the approximation can easily be estimated by means of the obvious inequality

$$t \exp[F(x, 0)] \leqslant \int_0^t \exp[F(x, \tau)]\, d\tau \leqslant t \exp[F(x, t)].$$

Thus the error does not exceed $\Delta I = 1 - \exp[F(x, 0) - F(x, t)]$, which is a quantity that is smaller in proportion to the extent to which it is true that $\exp[F(x, 0) - F(x, t)] \approx 1$.

In calculating the duration of the steady front, in analogy with [1], for the Bland number we have

$$\frac{t_0}{\Delta t} = \frac{4\,(c_h/c_p)\left(1 - c_h^2/c_p^2\right)^{-1}}{M\,(g\rho c v_0/\mu - \tau_*)\ln 19}. \tag{3.3}$$

The Bland number undergoes no significant change over a rather large range of velocities. The given formula was tested on experimental profiles [2]. Rather good agreement was found with experiment. Thus, for test firing 927 in [2], the thickness of the aluminum 6061-T6 target at which the profile becomes steady, is equal to 6.15 mm. According to formula (3.3), however, for this same test firing, with a stationary profile width of 60 nsec, the calculated target thickness is 6.13 mm.

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